IRREDUCIBLE REPRESENTATIONS OF CAYLEY–KLEIN ORTHOGONAL ALGEBRAS

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Abstract

Multidimensional contractions of irreducible representations of the Cayley–Klein orthogonal algebras in Gel'fand–Zetlin basis are considered. Contracted over different parameters, algebras can turn out to be isomorphic. In this case method of transitions describes the same reducible representations in different basises, say discrete and continuons ones.

1. Algebra $so\,(3;\boldsymbol{j})$

In spite that algebra $so\,(3)$ is isomorphic to algebra $u\,(2)$, we shall consider it separately, because algebras $so\,(3;\boldsymbol{j})$, $\boldsymbol{j}=(j_1,j_2)$, in contrast to $u\,(2;j_1)$, allow contraction over two parameters. Under transition from $so\,(3)$ to $so\,(3;\boldsymbol{j})$ generators are transformed as follows: $X_{01}=j_1X^*_{01}(\rightarrow)$, $X_{02}=j_1j_2X^*_{02}(\rightarrow)$, $X_{12}=j_2X^*_{12}(\rightarrow)$, and the only Casimir operator is transformed as $C_2(\boldsymbol{j})=j_1^2j_2^2C^*_2(\rightarrow)$ [1,2]. Gel'fand and Zetlin [3] have also found irreducible representations of algebra $so\,(3)$ in the basis, determined by the chain of subalgebras $so\,(3)\supset so\,(2)$; they are given by operators

$$
\begin{gathered}
X^*_{12}|m^*\rangle = im^*_{11}|m^*\rangle,\\
X^*_{01}|m^*\rangle = \frac{1}{2}\{\sqrt{(m^*_{12}-m^*_{11})(m^*_{12}+m^*_{11}+1)}|m^*_{11}+1\rangle-\\
-\sqrt{(m^*_{12}+m^*_{11})(m^*_{12}-m^*_{11}+1)}|m^*_{11}-1\rangle\},\\
X^*_{02}|m^*\rangle = \frac{i}{2}\{\sqrt{(m^*_{12}-m^*_{11})(m^*_{12}+m^*_{11}+1)}|m^*_{11}+1\rangle-\\
-\sqrt{(m^*_{12}+m^*_{11})(m^*_{12}-m^*_{11}+1)}|m^*_{11}-1\rangle\},
\end{gathered}
\tag{1}
$$

where the schemes $|\,m^*\rangle$, enumerating the elements of Gel'fand-Zetlin orthonormed basis, are $|\,m^*\rangle=\left|\begin{array}{c} m^*_{12}\\ m^*_{11}\end{array}\right\rangle$, $|m^*_{11}|\le m^*_{12}$ and components m^*_{11}, m^*_{12} are simultaneously either integer, or half-integer. Spectrum of Casimir operator is $C^*_2(m^*_{12})=m^*_{12}(m^*_{12}+1)$.

Component m^*_{11} is eigenvalue of operator X^*_{12}, for this reason its transformation is determined by transformation X^*_{12}, i.e. $m_{11}=j_2m^*_{11}$. The transformation of component m^*_{12} can be found from the requirement of determinacy and non-zero spectrum of Casimir operator $C_2(m_{12})=$ $=j_1^2j_2^2C^*_2(\rightarrow)=j_1^2j_2^2\dfrac{m_{12}}{j_1j_2}\left(\dfrac{m_{12}}{j_1j_2}+1\right)=m_{12}(m_{12}+j_1j_2)$ under contractions, i.e. $m_{12}=j_1j_2m^*_{12}$. Then we find from (1) operators of representa-

tion of algebra $so\,(3;\boldsymbol{j})$

$$
\begin{aligned}
X_{12}|m\rangle &= im_{11}|m\rangle,\\
X_{01}|m\rangle &= \frac{1}{2j_2}\big\{\sqrt{(m_{12}-j_1m_{11})(m_{12}+j_1m_{11}+j_1j_2)}|m_{11}+j_2\rangle-\\
&\quad-\sqrt{(m_{12}+j_1m_{11})(m_{12}-j_1m_{11}+j_1j_2)}|m_{11}-j_2\rangle\big\},\\
X_{02}|m\rangle &= \frac{i}{2}\big\{\sqrt{(m_{12}-j_1m_{11})(m_{12}+j_1m_{11}+j_1j_2)}|m_{11}+j_2\rangle+\\
&\quad+\sqrt{(m_{12}+j_1m_{11})(m_{12}-j_1m_{11}+j_1j_2)}|m_{11}-j_2\rangle\big\},
\end{aligned}
\tag{2}
$$

where the components of scheme $|\,m\rangle$ satisfy inequalities $|m_{11}|\le m_{12}/j_1$. Operators (2) satisfy commutation relations of algebra $so\,(3;\boldsymbol{j})$, which can be checked up straightforwardly [1]. The representation is irreducible. This can be established, as in the case of unitary algebras, by action of rising and lowering operators on vectors of major and minor weights.

For $j_1=\iota_1$ the relations (2) give irreducible representation of algebra $so\,(3;\iota_1,j_2)\equiv IO(2;j_2)=\{X_{01},X_{02}\}\niplus\{X_{12}\}$:

$$
\begin{aligned}
X_{12}|m\rangle &= im_{11}|m\rangle,\\
X_{01}|m\rangle &= \frac{m_{12}}{2j_2}(|m_{11}+j_2\rangle-|m_{11}-j_2\rangle),\\
X_{02}|m\rangle &= \frac{i}{2}m_{12}(|m_{11}+j_2\rangle+|m_{11}-j_2\rangle),
\end{aligned}
\tag{3}
$$

where m_{11} integer or half-integer, and $m_{12}\in\mathbb{R}$, $m_{12}\ge 0$. The eigenvalues of Casimir operator $C_2(\iota_1,j_2)$ are m_{12}^2.

For $j_2=\iota_2$ we obtain from (2) irreducible representation of algebra $so\,(3;j_1,\iota_2)=\{X_{02},X_{12}\}\niplus\{X_{01}\}$:

$$
\begin{aligned}
X_{12}|m\rangle = im_{11}|m\rangle,&\quad X_{02}|m\rangle = i\sqrt{m_{12}^2-j_1^2m_{11}^2}|m\rangle,\\
X_{01}|m\rangle = \sqrt{m_{12}^2-j_1^2m_{11}^2}&|m\rangle'_{11}-j_1^2\frac{m_{11}}{2\sqrt{m_{12}^2-j_1^2m_{11}^2}}|m\rangle,
\end{aligned}
\tag{4}
$$

where $m_{11},m_{12}\in\mathbb{R}$ and $|m_{11}|\le m_{12}$.

Ceasing to fix the coordinates x_0, x_1, x_2 in $\mathbb{R}_3(\boldsymbol{j})$, where group $SO(3;\boldsymbol{j})$ acts, we can easily prove isomorphism of algebras $so\,(3;\iota_1,1)$ and $so\,(3;1,\iota_2)$. Then (3) for $j_2 = 1$ and (4) for $j_1 = 1$ give description of irreducible representation of algebra $IO(2)$ in discrete and continuous basises, correspondingly, in infinite-dimensional space of representation. Discrete basis consists of eigenvectors of compact operator X_{12} with integer or half-integer eigenvalues $-\infty < m_{11} < \infty$. Continuous basis consists of generalized eigenvectors of noncompact operator X_{12}, which eigenvalues are $m_{11} \in \mathbb{R}$, $|m_{11}| \le m_{12}$. Celeghini [4–6] has considered close to ours approach to contraction and irreducible representations of orthogonal groups $SO(3)$, $SO(5)$, related with singular transformation of components of Gel'fand-Zetlin schemes.

Two-dimensional contraction $j_1 = \iota_1$, $j_2 = \iota_2$ gives irreducible representation of (two-dimensional) Galilean group (algebra) $so\,(3;\boldsymbol{\iota})$:

$$X_{01}|m\rangle = m_{12}|m\rangle'_{11}, \quad X_{12}|m\rangle = im_{11}|m\rangle, \\ X_{02}|m\rangle = im_{12}|m\rangle, \tag{5}$$

where $m_{12}, m_{11} \in \mathbb{R}$, $m_{12} \ge 0$, $-\infty < m_{11} < \infty$; $C_2(\boldsymbol{\iota}) = m_{12}^2$.

The result of action of generators on the derivative $|\,m\rangle'_{11} \equiv \dfrac{\partial}{\partial m_{11}}|\,m\rangle$ can be found, applying a generator to both sides of equation $|\,m\rangle'_{11} = \dfrac{2}{2\iota_2}(|\,m_{11}+\iota_2\rangle - |\,m_{11}-\iota_2\rangle)$. In particular, $X_{12}|\,m\rangle'_{11} = im_{11}\,|\,m\rangle'_{11} + i\,|\,m\rangle$.

2. Algebra $so\,(4;\boldsymbol{j})$

To determine an irreducible representation of algebra $so\,(4;\boldsymbol{j})$, $\boldsymbol{j} = (j_1, j_2, j_3)$, it is sufficient to give representation of generators X_{01}, X_{12}, X_{23}. Using the formulas from monograph [7], we can change indices of generators according to the rule: $4 \to 0$, $3 \to 1$, $2 \to 2$, $1 \to 3$. Then Gel'fand- Zetlin representation corresponds to the chain of subalgebras $so\,(4;\boldsymbol{j}) \supset so\,(3;j_2,j_3) \supset so\,(2;j_3)$, where $so\,(4;\boldsymbol{j}) = \{X_{\mu\nu},\ \mu<\nu,\ \mu,\nu = 0,1,2,3\}$; $so\,(3;j_2,j_3) = \{X_{\mu\nu},\ \mu<\nu,\ \mu,\nu = 1,2,3\}$; $so\,(2;j_3) = \{X_{23}\}$. Representation of generators X_{23}, X_{12} is given by (2), where

indices of generators and parameters $\boldsymbol{j}$ must be increased by 1, and scheme $|\,m\rangle = \left| \begin{matrix} m_{12} \\ m_{11} \end{matrix} \right\rangle$ has to be substituted for

$$|\,m\rangle = \left| \begin{matrix} m_{13} & & m_{23} \\ & m_{12} & \\ & m_{11} & \end{matrix} \right\rangle . \tag{6}$$

The rule of transformation for components m_{12}^*, m_{11}^* can be found by consideration of algebra $so\,(3;\boldsymbol{j})$: $m_{11}^* = j_3 m_{11}^*$, $m_{12} = j_2 j_3 m_{23}^*$, $|m_{11}| \leq$ $\leq m_{12}/j_2$. It remains to derive the transformation of components m_{13}^*, m_{23}^*, which determine irreducible representation. To this aim we consider spectrum of Casimir operators for algebra $so\,(4)$, found by A.M.Perelomov and V.S.Popov [8], A.N.Leznov, I.A.Malkin, V.I.Man'ko [9]:

$$C_2^* = m_{13}^*(m_{13}^* + 2) + m_{23}^{*2}, \quad C_2^{*'} = -(m_{13}^* + 1)m_{23}^*, \tag{7}$$

as well as the rule of transformation

$$\begin{aligned} C_{2p}(\boldsymbol{j}; X_{\mu\nu}) &= \left(\prod_{m=1}^{p-1} j_m^{2m} j_{n-m+1}^{2m} \prod_{l=p}^{n-p+1} j_l^{2p}\right) C_{2p}^*\left(X_{\mu\nu} \prod_{l=\mu+1}^{\nu} j_l^{-1}\right), \\ C_n'(\boldsymbol{j}; X_{\mu\nu}) &= \left(j_{\frac{n+1}{2}}^{\frac{n+1}{2}} \prod_{m=1}^{(n-1)/2} j_m^m j_{n-m+1}^m\right) C_n^{*'}\left(X_{\mu\nu} \prod_{l=\mu+1}^{\nu} j_l^{-1}\right). \end{aligned} \tag{8a}$$

for Casimir operators under transition from $so\,(4)$ to $so\,(4;\boldsymbol{j})$ [1]:

$$C_2(\boldsymbol{j}) = j_1^2 j_2^2 j_3^2 C_2^*(\rightarrow), \quad C_2'(\boldsymbol{j}) = j_1 j_2^2 j_3 C_2^{*'}(\rightarrow). \tag{8}$$

Requiring eigenvalues of operators $C_2(\boldsymbol{j})$ and $C_2'(\boldsymbol{j})$ to be determinate expressions under contractions, we get from (7), (8) for $C_2'(\boldsymbol{j})$

$$m_{12} m_{23} = j_1 j_2^2 j_3 m_{13}^* m_{23}^*. \tag{9}$$

This equation (if transformations of components m_{13}, m_{23} involve only the first powers of parameters $\boldsymbol{j}$) gives possible rules of transformation of these components.

Let us write down possible variants of transformations of irreducible representations of algebra $so\,(4)$ into representations of algebra $so\,(4;\boldsymbol{j})$ as well as transformed spectra of Casimir operators

$$
\begin{aligned}
&1)\quad m_{13} = j_1 j_2 m^*_{13}, \quad m_{23} = j_2 j_3 m^*_{23},\\
&C_2(\boldsymbol{j}) = j_3^2 m_{13}(m_{13} + 2j_1 j_2) + j_1^2 m_{23}^2,\\
&C'_2(\boldsymbol{j}) = -(m_{13} + j_1 j_2) m_{23};
\end{aligned}
\tag{10}
$$

$$
\begin{aligned}
&2)\quad m_{13} = j_2 m^*_{13}, \quad m_{23} = j_1 j_2 j_3 m^*_{23},\\
&C_2(\boldsymbol{j}) = m_{23}^2 + j_1^2 j_3^2 m_{13}(m_{13} + 2j_2),\\
&C'_2(\boldsymbol{j}) = -(m_{13} + j_2) m_{23};
\end{aligned}
\tag{11}
$$

$$
\begin{aligned}
&3)\quad m_{13} = j_1 j_2 j_3 m^*_{13}, \quad m_{23} = j_2 m^*_{23},\\
&C_2(\boldsymbol{j}) = m_{13}(m_{13} + 2j_1 j_2 j_3) + j_1^2 j_3^2 m_{23}^2,\\
&C'_2(\boldsymbol{j}) = -(m_{13} + j_1 j_2 j_3) m_{23}.
\end{aligned}
\tag{12}
$$

Considering (7), (8) only for operator $C_2(\boldsymbol{j})$, the following variants are admissible:

$$
\begin{aligned}
&4)\quad m_{13} = j_1 j_2 j_3 m^*_{13}, \quad m_{23} = m^*_{23},\\
&C_2(\boldsymbol{j}) = m_{13}(m_{13} + 2j_1 j_2 j_3) + j_1^2 j_2^2 j_3^2 m_{23}^2,\\
&C'_2(\boldsymbol{j}) = -j_2(m_{13} + j_1 j_2 j_3) m_{23};
\end{aligned}
\tag{13}
$$

$$
\begin{aligned}
&5)\quad m_{13} = m^*_{13}, \quad m_{23} = j_1 j_2 j_3 m^*_{23},\\
&C_2(\boldsymbol{j}) = m_{23}^2 + j_1^2 j_2^2 j_3^2 m_{13}(m_{13} + 2),\\
&C'_2(\boldsymbol{j}) = -j_2(m_{13} + 1) m_{23}
\end{aligned}
\tag{14}
$$

and other variants of transformations of components m^*_{13}, m^*_{23}, which include not all parameters $\boldsymbol{j}$, up to variant $m_{13} = m^*_{13}$, $m_{23} = m^*_{23}$.

Considering contraction $\boldsymbol{j} = \boldsymbol{\iota}$, we see that general nondegenerate (with non-zero eigenvalues of both Casimir operators) representations of contracted algebra $so\,(4;\boldsymbol{\iota})$ come out only in the case of transformations "2" and "3". In the case of transformations "1" we get $C_2(\boldsymbol{\iota}) = 0$, in the case of transformations "4", "5" – $C'_2(\boldsymbol{\iota}) = 0$. Under other transformations both Casimir operators have zero spectrum.

Let us consider variant "3". In this case components of scheme "1" satisfy inequalities

$$\frac{m_{13}}{j_1 j_3} \geq |m_{23}|, \quad \frac{m_{13}}{j_1 j_3} \geq \frac{m_{12}}{j_3} \geq |m_{23}|, \quad \frac{m_{12}}{j_2} \geq |m_{11}|, \tag{15}$$

interpreted for imaginary and dual values of parameters $\boldsymbol{j}$ according to the rules represented in [10]. Using (12) and rules of transformations for generators $X_{01} = j_1 X_{01}^*$, $X_{02} = j_1 j_2 X_{02}^*$, $X_{03} = j_1 j_2 j_3 X_{03}^*$, we find operators of irreducible representations of algebra $so\,(4; \boldsymbol{j})$:

$$\begin{gathered}
X_{01}|m\rangle = i m_{11}\beta|m\rangle - \frac{1}{j_2 j_3}\alpha\sqrt{m_{12}(m_{12}^2 - j_2^2 m_{11}^2)}|m_{12} - j_2 j_3\rangle + \\
+\frac{1}{j_2 j_3}\alpha(m_{12} + j_2 j_3)\sqrt{(m_{12} + j_2 j_3)^2 - j_2^2 m_{11}^2}|m_{12} + j_2 j_3\rangle, \\
X_{02}|m\rangle = \frac{i}{2}\beta[\sqrt{(m_{12} - j_2 m_{11})(m_{12} + j_2 m_{11} + j_2 j_3)}|m_{11} + j_3\rangle + \\
+\sqrt{(m_{12} - j_2 m_{11})(m_{12} - j_2 m_{11} + j_2 j_3)}|m_{11} - j_3\rangle] - \\
-\frac{1}{2j_3}\alpha(m_{12})\left[\sqrt{(m_{12} - j_2 m_{11})(m_{12} - j_2 m_{11} - j_2 j_3)}\left|\begin{matrix} m_{12} - j_2 j_3 \\ m_{11} + j_2 \end{matrix}\right\rangle - \right. \\
\left. -\sqrt{(m_{12} + j_2 m_{11})(m_{12} + j_2 m_{11} - j_2 j_3)}\left|\begin{matrix} m_{12} - j_2 j_3 \\ m_{11} - j_3 \end{matrix}\right\rangle\right] - \\
-\frac{1}{2j_3}\alpha(m_{12} + j_2 j_3)\times \qquad (16) \\
\times\left[\sqrt{(m_{12} + j_2 m_{11} + j_2 j_3)(m_{12} + j_2 m_{11} + 2j_2 j_3)}\left|\begin{matrix} m_{12} + j_2 j_3 \\ m_{11} + j_3 \end{matrix}\right\rangle - \right. \\
\left. -\sqrt{(m_{12} - j_2 m_{11} + j_2 j_3)(m_{12} - j_2 m_{11} + 2j_2 j_3)}\left|\begin{matrix} m_{12} + j_2 j_3 \\ m_{11} - j_3 \end{matrix}\right\rangle\right], \\
X_{03}|m\rangle = j_3\frac{1}{2}\beta[\sqrt{(m_{12} - j_2 m_{11})(m_{12} + j_2 m_{11} + j_2 j_3)}|m_{11} + \\
+ j_3\rangle - \sqrt{(m_{12} + j_2 m_{11})(m_{12} - j_2 m_{11} + j_2 j_3)}|m_{11} - j_3\rangle] +
\end{gathered}$$

$$
+\frac{i}{2}\alpha(m_{12})\left[\sqrt{(m_{12}-j_2m_{11})(m_{12}-j_2m_{11}-j_2j_3)}\left|\begin{matrix} m_{12}-j_2j_3 \\ m_{11}+j_3 \end{matrix}\right\rangle+\right.
$$

$$
\left.+\sqrt{(m_{12}+j_2m_{11})(m_{12}+j_2m_{11}-j_2j_3)}\left|\begin{matrix} m_{12}-j_2j_3 \\ m_{11}-j_3 \end{matrix}\right\rangle\right]+
$$

$$
+\frac{i}{2}\alpha(m_{12}+j_2j_3)\times
$$

$$
\times\left[\sqrt{(m_{12}+j_2m_{11}+j_2j_3)(m_{12}+j_2m_{11}+2j_2j_3)}\left|\begin{matrix} m_{12}+j_2j_3 \\ m_{11}+j_3 \end{matrix}\right\rangle+\right.
$$

$$
\left.+\sqrt{(m_{12}-j_2m_{11}+j_2j_3)(m_{12}-j_2m_{11}+2j_2j_3)}\left|\begin{matrix} m_{12}+j_2j_3 \\ m_{11}-j_3 \end{matrix}\right\rangle\right],
$$

$$
\alpha(m_{12})=\left\{\frac{[(m_{13}+j_1j_2j_3)^2-j_1^2m_{12}^2](m_{12}^2-j_3^2m_{23}^2)}{m_{12}^2(4m_{12}^2-j_2^2j_3^2)}\right\}^{\frac{1}{2}},
$$

$$
\beta=\frac{(m_{13}+j_1j_2j_3)m_{23}}{m_{12}(m_{12}+j_2j_3)}.
$$

Here we presented all generators of algebra $so\,(4;\boldsymbol{j})$ though, as it has been noticed, it is sufficient to give only X_{01}.

Initial finite-dimensional irreducible representation of algebra $so\,(4)$ is Hermitean. The representation (16) of algebra $so\,(4;\boldsymbol{j})$ is irreducible, but, in general non-Hermitean. To obtain Hermitean representation, it is necessary to impose on operators (16) the requirement of Hermiticity: $X^{\dagger}_{\mu\nu}=-X_{\mu\nu}$. It is difficult to find the restrictions implied by this requirement on the components of Gel'fand-Zetlin schemes. Therefore requirement of Hermiticity has to be checked up in any particular case for concrete values of parameters $\boldsymbol{j}$.

Considering variant (16), determined by (11), it turns out that components of scheme (6) satisfy inequalities

$$
m_{13}\geq\frac{|m_{23}|}{j_1j_3},\quad m_{13}\geq\frac{m_{12}}{j_3}\geq\frac{|m_{23}|}{j_1j_3},\quad \frac{m_{12}}{j_2}\geq|m_{11}|, \tag{17}
$$

operators X_{12}, X_{13}, X_{23} are described by (2), where indices of parameters and generators have to be increased by one, and operators X_{0k} ($k=1.2.3$) are given by (16), where functions β and $\alpha(m_{12})$ are substituted by

functions $\widetilde{\beta}$, $\widetilde{\alpha}(m_{12})$, which are as follows:

$$\widetilde{\alpha}(m_{12}) = \left\{ \frac{[j_3^2(m_{13}+j_2)^2 - m_{12}^2](j_1^2 m_{12}^2 - m_{23}^2)}{m_{12}^2(4m_{12}^2 - j_2^2 j_3^2)} \right\}^{\frac{1}{2}},$$
$$\widetilde{\beta} = \frac{(m_{13}+j_2)m_{23}}{m_{12}(m_{12}+j_2 j_3)}. \tag{18}$$

3. Contractions of representations of algebra $so\,(4;\boldsymbol{j})$

Let us consider representations of algebra $so\,(4;\iota_1,j_2,j_3) \equiv iso\,(3;j_2, j_3) = \{X_{0k}\} \supset\!\!\!\!\!\!+ \; so\,(3;j_2,j_3)$. Let components m_{13}, m_{23} are transformed according to (12), i.e. $k \equiv m_{13} = \iota_1 j_2 j_3 m_{13}^*$, $m_{23} = j_2 m_{23}^*$. Operators of representation are described by (16) where

$$\alpha(m_{12}) = k \left\{ \frac{m_{12}^2 - j_3^2 m_{13}^2}{m_{12}^2(4m_{12}^2 - j_2^2 j_3^2)} \right\}^{\frac{1}{2}}, \quad \beta = \frac{k m_{23}}{m_{12}(m_{12}+j_2 j_3)}. \tag{19}$$

¿From inequalities (15) for $j_2 = j_3 = 1$ we find $0 \le |m_{23}| < \infty$, $m_{12} \ge$ $\ge |m_{23}|$, $|m_{11}| \le m_{12}$, where m_{11}, m_{12}, $m_{23} \in \mathbb{Z}$; $k \in \mathbb{R}$ (the latter – from requirement of Hermiticity for X_{01}). Spectrum of Casimir operators comes out of (12): $C_2(\iota_1) = k^2$, $C_2'(\iota_1) = -k m_{23}$.

If components are transformed according to (11), i.e. $m_{13} = j_2 m_{13}^*$, $s \equiv m_{23} = \iota_1 j_2 j_3 m_{23}^*$ then α, β are substituted for

$$\widetilde{\alpha}(m_{12}) = is \left\{ \frac{j_3^2(m_{13}+j_2)^2 - m_{12}^2}{m_{12}^2(4m_{12}^2 - j_2^2 j_3^2)} \right\}^{\frac{1}{2}}, \quad \widetilde{\beta} = \frac{s(m_{13}+j_2)}{m_{12}(m_{12}+j_2 j_3)}. \tag{20}$$

Inequalities (17) for $j_2 = j_3 = 1$ determine $m_{13} \ge m_{12} \ge 0$, $|m_{11}| \le m_{12}$, $m_{11}, m_{12}, m_{13} \in \mathbb{Z}$, $s \in \mathbb{R}$. Spectrum of Casimir operators comes out of (11): $C_2(\iota_1) = s^2$, $C_2'(\iota_1) = -s m_{13}$.

For algebra $so\,(4;j_1,j_2,\iota_3) = T_3 \supset\!\!\!\!\!\!+ \; so\,(3;j_1,j_2)$, where $T_3 = \{X_{03}, X_{13}, X_{23}\}$ under transformation (12), i.e. $k \equiv m_{13} = j_1 j_2 \iota_3 m_{13}^*$, $m_{23} =$

$= j_2 m_{23}^*$, $p \equiv m_{12} = j_2 \iota_3 m_{12}^*$, $q \equiv m_{11} = \iota_3 m_{11}^*$, relations (16) determine operators of irreducible representations:

$$
\begin{gathered}
X_{01}|m\rangle = i\frac{kqm_{23}}{p^2}|m\rangle + \\
+\frac{1}{2p}\sqrt{(p^2 - j_2^2 q^2)(k^2 - j_1^2 p^2)}\,\{\, 2|m\rangle'_p + \frac{p}{p^2 - j_2^2 q^2}|m\rangle - \\
-\frac{k^2}{p(k^2 - j_1^2 p^2)}|m\rangle \,\}\,, \quad X_{03}|m\rangle = i\sqrt{k^2 - j_1^2 p^2}|m\rangle, \\
X_{02}|m\rangle = i\frac{km_{23}}{p^2}\sqrt{p^2 - j_2^2 q^2}|m\rangle - \\
-\frac{1}{2p}\sqrt{k^2 - j_1^2 p^2}\,\{\, 2p|m\rangle'_q + 2j_2^2 q|m\rangle'_p - \\
-j_2^2\frac{qk^2}{p(k^2 - j_1^2 p^2)}|m\rangle \,\}\,, \quad |m\rangle = \left| \begin{array}{ccc} k & & m_{23} \\ & p & \\ & q & \end{array} \right\rangle .
\end{gathered}
\tag{21}
$$

The rest operators are given by (4) with obvious modifications. Spectrum of Casimir operators is $C_2(\iota_3) = k^2$, $C'_2(\iota_3) = -km_{23}$.

Inequalities (15) for $j_1 = j_2 = 1$ imply $0 \le |m_{23}| < \infty$, $k \ge p \ge 0$, $|q| \le p$, $m_{23} \in \mathbb{Z}$, $k, p, q \in \mathbb{R}$.

For transformation of components (11), i.e. $m_{13} = j_2 m_{13}^*$, $s \equiv$ $\equiv m_{23} = j_1 j_2 \iota_3 m_{23}^*$, $p \equiv m_{12} = j_2 \iota_3 m_{12}^*$, $q \equiv m_{11} = \iota_3 m_{13}^*$, irreducible representation of algebra $so\,(4; j_1, j_2, \iota_3)$ is described by operators

$$
\begin{gathered}
X_{01}|\widetilde{m}\rangle = i\frac{sq(m_{13} + j_2)}{p^2}|\widetilde{m}\rangle + \frac{i}{2p}\sqrt{(p^2 - j_2^2 q^2)(j_1^2 p^2 - s^2)}\times \\
\times\left\{2|\widetilde{m}\rangle'_p + \frac{s^2}{p(j_1^2 p^2 - s^2)}|\widetilde{m}\rangle + \frac{p}{p^2 - j_2^2 q^2}|\widetilde{m}\rangle\right\}, \\
X_{02}|\widetilde{m}\rangle = i\frac{s(m_{13} + j_2)}{p^2}\sqrt{p^2 - j_2^2 q^2}|\widetilde{m}\rangle -
\end{gathered}
\tag{22}
$$

$$-\frac{i}{2p}\sqrt{j_1^2p^2-s^2}\,\{\; 2p|\widetilde{m}\rangle'_q+2j_2^2q|\widetilde{m}\rangle'_p+j_2^2\frac{s^2q}{p(j_1^2p^2-s^2)}|\widetilde{m}\rangle\;\},$$

$$X_{03}|\widetilde{m}\rangle=-\sqrt{j_1^2p^2-s^2}|\widetilde{m}\rangle,\quad |\widetilde{m}\rangle=\left|\begin{matrix} m_{13} & & s \\ & p & \\ & q & \end{matrix}\right\rangle.$$

Components of scheme $|\,\widetilde{m}\rangle$ satisfy inequalities implied by (11) for $j_1 = = j_2 = 1$: $m_{13}\geq 0$, $p\geq |s|$, $|q|\leq p$, $m_{13}\in\mathbb{Z}$, $p,q,s\in\mathbb{R}$. Spectrum of Casimir operators is as follows: $C_2(\iota_3)=s^2$, $C'_2(\iota_3)=-s\,(m_{13}+j_2)$. It follows from (4) and (22) that generators X_{13}, X_{23},$X_{03}\in T_3$ are diagonal in continuous basis $|\,\widetilde{m}\rangle$.

Rejecting to fix coordinate axes, we notice that algebra $so\,(4;\iota_1,1,1)$ is isomorphic to algebra $so\,(4;1,1,\iota_3)$, and both these algebras are isomorphic to inhomogeneous algebra $iso\,(3)$. Isomorphism can be established by putting generator $X_{\mu\nu}$, $\mu<\nu$ in correspondence with generator $X_{3-\nu,3-\mu}$ of another algebra. Then operators (16) and (19) determine irreducible representation of algebra $iso\,(3)$ in discrete basis corresponding to the chain of subalgebras $iso\,(3)\supset so\,(3)\supset so\,(2)$, and operators (21) and (4) describe the same representation in continuous basis, corresponding to the chain $iso\,(3)\supset so\,(3;1,\iota_3)\supset so\,(2;\iota_3)$. The same assertion is valid for another variant of transition from representation of algebra $so\,(4)$ to representations of algebra $so\,(4;\boldsymbol{j})$, which brings to (16), (20) and (4), (22).

It is worth of noticing that contractions of representations give another way of constructing the irreducible representations of algebras (groups) with the structure of semidirect sum (product).

Operators of irreducible representation of algebra $so\,(4;j_1,\iota_2,j_3)$ come out of (16) for $j_2=\iota_2$ and can be written as follows:

$$X_{01}|m\rangle=i\frac{ksm_{11}}{p^2}|m\rangle+f(k,p,s)\left(2|m\rangle'_p+j_3^2\frac{s^2}{p(p^2-j_3^2s^2)}|m\rangle-\right.$$

$$\left.-j_1^2\frac{p^2}{k^2-j_1^2p^2}|m\rangle\;\right),$$

$$X_{02}|m\rangle=i\frac{ks}{2p}(|m_{11}+j_3\rangle+|m_{11}-j_3\rangle)-$$

$$
\begin{gathered}
-\frac{1}{j_3} f(k,p,s)(|m_{11}+j_3\rangle - |m_{11}-j_3\rangle), \\
X_{03}|m\rangle = i\frac{ks}{2p}(|m_{11}-j_3\rangle - |m_{11}-j_3\rangle) + if(k,p,s)\times \\
\times(|m_{11}+j_3\rangle + |m_{11}-j_3\rangle), \\
f(k,s,p) = \frac{1}{2p}\sqrt{(k^2-j_1^2p^2)(p^2-j_3^2s^2)}, \\
|m\rangle = \left| \begin{matrix} k & & s \\ & p & \\ & m_{11} & \end{matrix} \right\rangle .
\end{gathered}
\tag{23}
$$

Components of scheme $|\,m\rangle$ satisfy inequalities: $k \geq |s|$, $-\infty < s < \infty$, $k \geq p \geq |s|$, $-\infty < m_{11} < \infty$, $k,s,p \in \mathbb{R}$, $m_{11} \in \mathbb{Z}$, if $j_1 = j_3 = 1$. Spectrum of Casimir operators is as follows: $C_2(\iota_2) = k^2 + j_1^2 j_3^2 s^2$, $C_2'(\iota_2) = -ks$.

For algebra $so\,(4;\iota_1,\iota_2,j_3)$ operators of irreducible representation are given by (16) for $j_1 = \iota_1$, $j_2 = \iota_2$, i.e.

$$
\begin{gathered}
X_{01}|m\rangle = i\frac{ksm_{11}}{p^2}|m\rangle + \frac{k}{2p}\sqrt{p^2-j_3^2s^2}\times \\
\times\left(2|m\rangle_p' + j_3^2\frac{s^2}{p(p^2-j_3^2s^2)}|m\rangle\right), \\
X_{02}|m\rangle = i\frac{ks}{2p}(|m_{11}+j_3\rangle + |m_{11}-j_3\rangle) - \\
-\frac{1}{j_3}\frac{k}{2p}\sqrt{p^2-j_3^2s^2}(|m_{11}+j_3\rangle - |m_{11}-j_3\rangle), \\
X_{03}|m\rangle = i\frac{ks}{2p}(|m_{11}+j_3\rangle - |m_{11}-j_3\rangle) + \\
+\frac{ik}{2p}\sqrt{p^2-j_3^2s^2}(|m_{11}+j_3\rangle + |m_{11}-j_3\rangle).
\end{gathered}
\tag{24}
$$

Components of scheme $|\,m\rangle$ for $j_3 = 1$ satisfy inequalities: $k \geq 0$, $-\infty < s < \infty$, $p \geq |s|$, $-\infty < m_{11} < \infty$, $k,s,p \in \mathbb{R}$, $m_{11} \in \mathbb{Z}$. Spectrum of Casimir operators is as follows: $C_2(\iota_1,\iota_2) = k^2$, $C_2'(\iota_1,\iota_2) = -ks$.

Irreducible representation of algebra $so\,(4; j_1, \iota_2, \iota_3)$ is given by

$$
\begin{aligned}
X_{01}|m\rangle &= i\frac{ksq}{p}|m\rangle + \frac{1}{2}\sqrt{k^2 - j_1^2 p^2}\times \\
&\times \left(2|m\rangle'_p - j_1^2 \frac{p^2}{(k^2 - j_1^2 p^2)}|m\rangle\right), \\
X_{02}|m\rangle &= i\frac{ks}{p}|m\rangle - \sqrt{k^2 - j_1^2 p^2}|m\rangle'_q, \\
X_{03}|m\rangle &= i\sqrt{k^2 - j_1^2 p^2}|m\rangle, \quad |m\rangle = \left|\begin{matrix} k & & s \\ & p & \\ & q & \end{matrix}\right\rangle .
\end{aligned} \tag{25}
$$

which come out of (16) for $j_2 = \iota_2$, $j_3 = \iota_3$. For $j_1 = 1$ the components of scheme $|\,m\rangle$ satisfy inequalities: $k \geq 0$, $-\infty < s < \infty$, $k \geq p \geq 0$, $-\infty < q < \infty$, $k, s, p, q \in \mathbb{R}$. Spectrum of Casimir operators are the same as in (24).

Algebra $so\,(4; \iota_1, \iota_2, j_3) = \widetilde{T}_5 \Supset so\,(2; j_3)$, where $so\,(2; j_3) = \{X_{23}\}$, is isomorphic to algebra $so\,(4; j_1, \iota_2, \iota_3) = T'_5 \Supset so\,(2; j_1)$, $so\,(2; j_1) = \{X_{01}\}$, and they both are isomorphic to algebra $a = T_5 \Supset K$, where T_5 is nilpotent radical, and K is one-dimensional component subalgebra. Therefore (24) determine irreducible representation of algebra a in basis, corresponding to the chain of subalgebras $so\,(4; \iota_1, \iota_2, j_3) \supset so\,(3; \iota_2, j_3)$ *supset* $so\,(2; j_3)$, where $so\,(2; j_3) = \{X_{23}\}$ is compact subalgebra with discrete eigenvalues m_{11}, and (25) describe the same representation of algebra a in continuous basis, determined by the chain $so\,(4; j_1, \iota_2, \iota_3) \supset so\,(3; \iota_2, \iota_3) \supset so\,(2; \iota_3)$; where $(so\,(2; \iota_3) = \{X_{23}\})$ is already noncompact generator with continuous eigenvalues q.

For $j_1 = \iota_1$, $j_3 = \iota_3$ formulas (16) give irreducible representation of algebra $so\,(4; \iota_1, j_2, \iota_3)$:

$$
\begin{aligned}
X_{01}|m\rangle &= i\frac{kqm_{23}}{p^2}|m\rangle + \frac{k}{2p}\sqrt{p^2 - j_2^2 q^2}\times \\
&\times \left(2|m\rangle'_p + j_2^2 \frac{q^2}{p(p^2 - j_2^2 q^2)}|m\rangle\right),
\end{aligned}
$$

$$X_{02}|m\rangle = i\frac{km_{23}}{p^2}\sqrt{p^2 - j_2^2 q^2}|m\rangle -$$
$$-\frac{k}{2p}\left(2j_2^2 q|m\rangle'_p + 2p|m\rangle'_q - j_2^2\frac{q}{p}|m\rangle\right), \tag{26}$$
$$X_{03}|m\rangle = ik|m\rangle, \quad |m\rangle = \left|\begin{matrix} k & & m_{23} \\ & p & \\ & q & \end{matrix}\right\rangle.$$

Components of scheme $|\,m\rangle$ for $j_2 = 1$ satisfy following inequalities: $k \geq 0$, $-\infty < m_{23} < \infty$, $p \geq 0$, $|q| \leq p$, $k, p, q \in \mathbb{R}$, $m_{23} \in \mathbb{Z}$. Spectrum of Casimir operators are as follows: $C_2(\iota_1, \iota_3) = k^2$, $C'_2(\iota_1, \iota_3) = -km_{23}$. Three-dimensional contraction $\boldsymbol{j} = \boldsymbol{\iota}$ turns (16) into irreducible representation of maximally contracted algebra $so\,(4; \boldsymbol{\iota})$, described by operators

$$X_{01}|m\rangle = i\frac{ksq}{p^2}|m\rangle + k|m\rangle'_p \quad X_{03}|m\rangle = ik|m\rangle,$$
$$X_{02}|m\rangle = i\frac{ks}{p}|m\rangle - k|m\rangle'_q, \quad |m\rangle = \left|\begin{matrix} k & & s \\ & p & \\ & q & \end{matrix}\right\rangle. \tag{27}$$

with eigenvalues of Casimir operators: $C_2(\boldsymbol{\iota}) = k^2$, $C'_2(\boldsymbol{\iota}) = -ks$. Components of scheme $|\,m\rangle$ are real, continuous and satisfy following inequalities: $k \geq 0$, $-\infty < s < \infty$, $p \geq 0$, $-\infty < q < \infty$.

4. Algebra $so\,(n; \boldsymbol{j})$

In [10] we have discussed in detail possible variants of transformations of irreducible representation under transition from algebra $so\,(4)$ to algebra $so\,(4; \boldsymbol{j})$. For orthogonal algebras of arbitrary dimension we shall not consider all possible variants, but dwell on the (basic) variant, in which number of parameters $\boldsymbol{j}$, on which components some row of Gel'fand-Zetlin schemes, diminishes with increasing of component number in this row. The transformation of components under transition from algebra $so\,(n)$ to $so\,(n; \boldsymbol{j})$ can be found from the rule of transformation for Casimir operators. Because orthogonal algebras of even and odd dimensions have different sets of Casimir operators, we shall discuss these cases separately.

Algebra $so\,(2k+2;\boldsymbol{j})$, $\boldsymbol{j}=(j_1,\ldots,j_{2k+1})$, is characterized by a set of $k+1$ invariant operators [1]

$$C_{2p}(\boldsymbol{j})=\prod_{s=1}^{p-1} j_s^{2s} j_{2(k+1)-s}^{2s} \prod_{l-p}^{2(k+1)-p} j_l^{2p} C_{2p}^*(\to), \quad p=1,2,\ldots,k,$$
$$C'_{k+1}(\boldsymbol{j})=j_{k+1}^{k+1}\prod_{l=1}^{k} j_l^l j_{2(k+1)-l}^l C_{k+1}^{*'}(\to). \tag{28}$$

Gel'fand-Zetlin scheme for algebra $so\,(2k+2)$ is as follows:

$$|m\rangle^* = \left| \begin{array}{ccccc} m^*_{1,2k+1} & m^*_{2,2k+1} & \ldots & m^*_{k,2k+1} & m^*_{k+1,2k+1} \\ & m^*_{1,2k} & \ldots & m^*_{k,2k} & \\ & m^*_{1,2k-1} & \ldots & m^*_{k,2k-1} & \\ & m^*_{1,2k-2} & \ldots & m^*_{k-1,2k-2} & \\ & m^*_{1,2k-3} & \ldots & m^*_{k-1,2k-3} & \\ \multicolumn{5}{c}{\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots} \\ & & m^*_{12} & & \\ & & m^*_{11} & & \end{array} \right\rangle . \tag{29}$$

Irreducible representation as well as spectrum of Casimir operators on this representation are completely determined by components $m^*_{p,2k+1}$ of the upper row of scheme (29). Component $m^*_{1,2k+1}$ enters spectrum of Casimir operator C_2^* quadratically, and due to this fact in the basic variant it is transformed according to the rule $m_{1,2k+1}=m^*_{1,2k+1}\prod_{l=1}^{2k+1} j_l$. The transformation of the component $m^*_{p,2k+1}$ coincides with transformation of algebraic quantity $\{C^*_{2p}/C^*_{2(p-1)}\}^{1/2}$:

$$m_{p,2k+1}=m^*_{p,2k+1}\prod_{l=p}^{2(k+1)-p} j_l \equiv m^*_{p,2k+1}J_{p,2k+1}. \tag{30}$$

Component $m^*_{k+1,2k+1}$ is transformed in the same way as the ratio $C^{*'}_{k+1}/$ $/\{C^*_{2k}\}^{1/2}$, i.e. $m_{k+1,2k+1}=j_{k+1}m^*_{k+1,2k+1}$. This relation comes out of

(30) for $p = k+1$ as well as relation for transformation of component $m^*_{1,2k+1}$ that comes out of (30) for $p = 1$. Thus, all components of major weight (the upper row) are transformed according to (30). Inequalities, which governed them in classical case, turns into

$$\begin{gathered}\frac{m_{p,2k+1}}{J_{p,2k+1}} \geq \frac{m_{p+1,2k+1}}{J_{p+1,2k+1}}, \\ \frac{m_{k,2k+1}}{J_{k,2k+1}} \geq \frac{|m_{k+1,2k+1}|}{J_{k+1,2k+1}}, \\ p = 1,2,\ldots,k-1.\end{gathered} \tag{31}$$

Similarly, transformation of components of the row with number $2k$ of scheme (29) is determined by the rules of transformations for Casimir operators of subalgebra $so\,(2k+1; j_2, j_3, \ldots, j_{2k+1})$ and given by (30), in which product of parameters j_l starts with $p+1$. In general the rule of transformation for components of scheme (29) can be easily found and turns out to be as follows:

$$\begin{gathered}m_{p,2s+1} = J_{p,2s+1} m^*_{p,2s+1}, \quad J_{p,2s+1} = \prod_{l=p+2(k-s)}^{2(k+1)-p} j_l, \\ s = 0,1,\ldots,k, \quad p = 1,2,\ldots,s+1, \\ m_{p,2s} = J_{p,2s} m^*_{p,2s}, \quad J_{p,2s} = \prod_{l=p+2(k-s)+1}^{2(k+1)-p} j_l, \\ s = 1,2,\ldots,k, \quad p = 1,2,\ldots,s.\end{gathered} \tag{32}$$

The transformed components are governed by inequalities

$$\begin{gathered}\frac{m_{p,2s+1}}{J_{p,2s+1}} \geq \frac{m_{p,2s}}{J_{p,2s}} \geq \frac{m_{p+1,2s+1}}{J_{p+1,2s+1}}, \quad p = 1,2,\ldots,s-1, \\ \frac{m_{s,2s+1}}{J_{s,2s+1}} \geq \frac{m_{s,2s}}{J_{s,2s}} \geq \frac{|m_{s+1,2s+1}|}{J_{s+1,2s+1}}, \\ \frac{m_{p,2s}}{J_{p,2s}} \geq \frac{m_{p,2s-1}}{J_{p,2s-1}} \geq \frac{m_{p+1,2s}}{J_{p+1,2s}}, \quad p = 1,2,\ldots,s-1, \\ \frac{m_{s,2s}}{J_{s,2s}} \geq \frac{m_{s,2s-1}}{J_{s,2s-1}} \geq -\frac{m_{s,2s}}{J_{s,2s}},\end{gathered} \tag{33}$$

which for dual and imaginary values of parameters $\boldsymbol{j}$ are interpreted according to the rules described in [10].

Action of the whole algebra $so\,(2k+2;\boldsymbol{j})$ can be reproduced by giving the action of generators $X_{2(k-s)+1,2(k-s+1)}$, $s = 1,2,\ldots,k$, $X_{2(k-s),2(k-s)+1}$, $s = 0,1,\ldots,k-1$. Transforming the expressions for these generators, which can be found in [7], we obtain

$$X_{2(k-s)+1,2(k-s+1)}|m\rangle = \sum_{p=1}^{s} \frac{1}{J_{p,2s-1}} \{A(m_{p,2s-1})|m_{p,2s-1}+$$

$$+J_{p,2s-1}\rangle - A(m_{p,2s-1}-J_{p,2s-1})|m_{p,2s-1}-J_{p,2s-1}\rangle\},$$

$$X_{2(k-s),2(k-s)+1}|m\rangle = iC_{2s}|m\rangle + \sum_{p=1}^{s} \frac{1}{J_{p,2s}} \{B(m_{p,2s})|m_{p,2s}+$$

$$+J_{p,2s}\rangle - B(m_{p,2s}-J_{p,2s})|m_{p,2s}+J_{p,2s}\rangle\},$$

$$C_{2s} = \prod_{p=1}^{s} l_{p,2s-1} \prod_{p=1}^{s+1} l_{p,2s+1} \prod_{p=1}^{s} l_{p,2s}^{-1}(l_{p,2s}-J_{p,2s})^{-1},$$

$$B(m_{p,2s}) = \Bigg\{ \prod_{r=1}^{p-1} (l_{r,2s-1}^2 - l_{p,2s}^2 a_{r,p,s}^2) \prod_{r=p}^{s} (l_{r,2s-1}^2 a_{r,p,s}^{-2} - l_{p,2s}^2) \times \tag{34}$$

$$\times \prod_{r=1}^{p} (l_{r,2s+1}^2 - l_{p,2s}^2 b_{r,p,s}^2) \prod_{r=p+1}^{s+1} (l_{r,2s+1}^2 b_{r,p,s}^{-2} - l_{p,2s}^2) \Bigg\}^{\frac{1}{2}} \times$$

$$\times \Bigg\{ l_{p,2s}^2 (4l_{p,2s}^2 - J_{p,2s}^2) \prod_{r=1}^{p-1} (l_{r,2s}^2 - l_{p,2s}^2 c_{r,p,s}^2)[(l_{r,2s}-J_{r,2s})^2 - l_{p,2s} c_{r,p,s}^2] \times$$

$$\times \prod_{r=p+1}^{s} (l_{r,2s}^2 c_{r,p,s}^{-2} - l_{p,2s}^2)[(l_{r,2s}-J_{r,2s})^2 c_{r,p,s}^{-2} - l_{p,2s}^2] \Bigg\}^{-\frac{1}{2}},$$

$$A(m_{p,2s-1}) = \frac{1}{2} \Bigg\{ \prod_{r=1}^{p-1} (l_{r,2s-2} - l_{p,2s-1} a_{r,p,s-1/2} - J_{r,2s-2}) \times$$

$$\times (l_{r,2s-2} + l_{p,2s-1} a_{r,p,s-1/2}) \prod_{r=p}^{s-1} (l_{p,2s-2} a_{r,p,s-1/2}^{-1} - l_{p,2s-1} -$$

$$-J_{p,2s-1})(l_{r,2s-2}a^{-1}_{r,p,s-1/2}+l_{p,2s-1})\times$$
$$\times\prod_{r=1}^{p}(l_{r,2s}-l_{p,2s-1}b_{r,p,s-1/2}-J_{r,2s})(l_{r,2s}+$$
$$+l_{p,2s-1}b_{r,p,s-1/2}\prod_{r=p+1}^{s}(l_{r,2s}b^{-1}_{r,p,s-1/2}-l_{p,2s-1}-J_{p,2s-1})\times$$
$$\times(l_{r,2s}b^{-1}_{r,p,s-1/2}+l_{p,2s-1})\Big\}^{\frac{1}{2}}\Big\{\prod_{r=p+1}^{s}(l^2_{r,2s-1}c^{-2}_{r,p,s-1/2}-l^2_{p,2s-1})\times$$
$$\times[l^2_{r,2s-1}-(l_{r,2s-1}+J_{p,2s-1})c^2_{r,p,s-1/2}]\prod_{r=p+1}^{s}l^2_{r,2s-1}c^{-2}_{r,p,s-1/2}-$$
$$-l^2_{p,2s-1})[l^2_{r,2s-1}c^{-2}_{r,p,s-1/2}-(l_{p,2s-1}+J_{p,2s-1})^2]\Big\}^{-\frac{1}{2}},$$
$$a_{r,p,s}=J_{r,2s-1}/J_{p,2s},\quad b_{r,p,s}=J_{r,2s+1}/J_{p,2s},$$
$$c_{r,p,s}=J_{r,2s}/J_{p,2s},\quad l_{p,2s}=m_{p,2s}+(s-p+1)J_{p,2s}.$$

For algebra $so\,(2k+1)$ Gel'fand-Zetlin scheme $|\,m^*\rangle$ coincides with (29) with deleted row with number $2k+1$. The upper row, determining the components of major weight, now is the row with number $2k$; its components satisfy inequalities $m^*_{1,2k}\geq m^*_{2,2k}\geq\cdots\geq m^*_{k,2k}\geq 0$. Under transition from classical algebra $so\,(2k+1)$ to algebras $so\,(2k+1;\boldsymbol{j})$, $\boldsymbol{j}=(j_1,\ldots,j_{2k})$ the components of scheme $|\,m\rangle$ are transformed as follows:

$$m_{p,2s}=m^*_{p,2s}J_{p,2s},\quad J_{p,2s}=\prod_{l=p+2(k-s)}^{2k+1-p}j_l,$$
$$m_{p,2s-1}=m^*_{p,2s-1}J_{p,2s-1},\quad J_{p,2s-1}=\prod_{l=p+2(k-s)+1}^{2k+1-p}j_l, \tag{35}$$
$$s=1,2,\ldots,k,\quad p=1,2,\ldots,s.$$

Let us draw attention to the fact that the lower limits in the product, determining $J_{p,2s}$, $J_{p,2s-1}$, have changed in comparison with (32). This

is due to the diminishing of the number of parameters $\boldsymbol{j}$ by one in the case of algebra $so\,(2k+1;\boldsymbol{j})$ in comparison with algebra $so\,(2(k+1);\boldsymbol{j})$. Components of the upper row in scheme $|\,m\rangle$ satisfy inequalities

$$\frac{m_{1,2k}}{J_{1,2k}} \geq \frac{m_{2,2k}}{J_{2,2k}} \geq \cdots \geq \frac{m_{k-1,2k}}{J_{k-1,2k}} \geq \frac{m_{k,2k}}{J_{k,2k}} \geq 0, \tag{36}$$

and the other components are governed by inequalities (33), which parameters $J_{p,2s}$, $J_{p,2s-1}$ are defined according to (35). Operators of irreducible representation of algebra $so\,(2k+1;\boldsymbol{j})$ are given by (34) with parameters from (35).

Operators (34) satisfy commutation relations of algebra $so\,(n;\boldsymbol{j})$. This can be checked up by straightforward calculations as well. The irreducibility of representation is implied by consideration of action of rising and lowering operators on vectors of major and minor weights and nonzero result of this action for dual and imaginary values of parameters $\boldsymbol{j}$. Though the initial representation of algebra $so\,(n)$ is Hermitean, the representation (34), in general is not such. Requiring the fulfillment of condition $X^{\dagger}_{\mu\nu} = -X_{\mu\nu}$, we find those values of transformed components of Gel'fand-Zetlin scheme, for which representation (34) will be Hermitean.

It is worth of noticing, at last, that for imaginary values of parameters $\boldsymbol{j}$ the relations (34) give irreducible representations of pseudoorthogonal algebras of different signature.

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